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Painlevé analysis of the non-linear Schrödinger family of equations

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Abstract. In this paper we apply the Painlevé tests to the generalised derivative non-linear Schrödinger equation,

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auu^*u_x + \mathrm{i}bu^2u_x^* + cu^3u^{*2}$$

where u^* denotes the complex conjugate of u, and a, b and c are real constants, to determine under what conditions the equation might be completely integrable. It is shown that, apart from a trivial multiplicative factor, this equation possesses the Painlevé property for partial differential equations as formulated by Weiss, Tabor and Carnevale only if $c = \frac{1}{4}b(2b-a)$. When this relation holds, this is equivalent under a gauge transformation to the derivative non-linear Schrödinger equation (DNLS) of Kaup and Newell, which is known to be completely integrable, or else to a linear equation. In addition, we consider a generalisation of the mixed non-linear Schrödinger equation of Wadati *et al* obtained by adding a term du^2u^* to the right-hand side of the equation and once again find that the only cases which possess the Painlevé property are transformable to the derivative non-linear Schrödinger equation or the original non-linear Schrödinger equation (NLS), which is known to be completely integrable. In the final section, we study the singularities of the Wadati-Konno-Ichikawa (WKI) equation and a new equation of Dodd and Fordy, and we conjecture that the latter is non-integrable. We also take a closer look at Ishimori's Bäcklund transformation relating the WKI equation to NLS and find that it factorises into a simple easily invertible transformation relating WKI to DNLS and a known Miura transformation relating DNLS to NLS. As an application we derive a new three-parameter Lie group of transformations admitted by WKI.

1. Introduction

Several very important, physically interesting, non-linear partial differential equations (PDE) are solvable by *inverse scattering*, such as the Korteweg-de Vries equation, the non-linear Schrödinger equation (NLS), the Boussinesq equation, the sine-Gordon equation and the Kadomtsev-Petviashvili equation (cf Ablowitz and Segur 1981). This method, in effect, reduces the solution of the non-linear equation to that of a linear integral equation and the PDE is then said to be *completely integrable.*

The Painlevé conjecture, as formulated by Ablowitz *et al* (1978, 1980a) and Hastings and McLeod (1980), asserts that every ordinary differential equation (ODE) which arises as a similarity reduction of a PDE solvable by inverse scattering is of Painlevé type, i.e. it has no movable singularities except poles, perhaps after a transformation of variables. Ablowitz *et al* (1980a) and McLeod and Olver (1983) have given proofs of the conjecture under certain restrictions. Subsequently, Weiss *et al* (1983) have

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defined the Painlevé property for PDE and developed a method for testing for a common particular type of movable singularity, without having to study any similarity reductions (which may not exist anyway). A PDE is said to *possess the Painlevé property* if the only singularities of the general integral which can live on arbitrary non-characteristic ('movable') hypersurfaces are poles (all variables are presumed complexified). These Painlevé tests have proved to be a useful criterion for the identification of completely integrable PDE.

The purpose of §§ 2 and 3 of this paper is to determine, using the Painlevé tests, for which real values of the parameters a, b and c the generalised derivative non-linear Schrödinger equation (GDNLS),

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auu^* u_x + \mathrm{i}bu^2 u_x^* + cu^3 u^{*2} \tag{1.1}$$

where u^* denotes the complex conjugate of u, is completely integrable. This equation is a generalisation of both the derivative non-linear Schrödinger (DNLS) equation considered by Kaup and Newell (1978)

$$\mathrm{i}u_t = u_{xx} + 2\mathrm{i}buu^* u_x + \mathrm{i}bu^2 u_x^* \tag{1.2}$$

which we shall call DNLSI, and that considered by Chen *et al* (1979):

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auu^* u_x \tag{1.3}$$

which we shall call DNLSII (see equation (2.13) below for a third case). Kaup and Newell (1978) solved the initial value problem for DNLSI under the restriction $u(x, t) \to 0$ as $|x| \to \infty$ using an inverse scattering formalism of Zakharov–Shabat type. This has subsequently been generalised to $u(x, t) \to$ constant (Kawata and Inoue 1978) and to $u(x, t) \to$ (constant) $\exp[\mathrm{i}(kx - \omega t)]$ (Kawata *et al* 1980). While DNLSII does not appear explicitly in Kaup and Newell (1978), their variables Q and R solve (1.3) with $Q = u^*$, $R = u$, $a = 1$. Chen *et al* (1979) and Dodd and Fordy (1983) write down associated linear problems for equation (1.3) but do not attempt to solve the equation.

In § 3 we show that the Painlevé tests suggest that a necessary condition for the GDNLS equation (1.1) to be completely integrable is that

$$c = \tfrac{1}{4}b(2b - a) \tag{1.4}$$

and then under this condition (1.1) becomes

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auu^* u_x + \mathrm{i}bu^2 u_x^* + \tfrac{1}{4}b(2b - a)u^3 u^{*2} \tag{1.5}$$

which Kundu (1984) calls the 'higher-order non-linear Schrödinger equation', a term which we would rather save for generalisations to higher orders of differentiation. We use the word 'suggest' in this context because we are aware that a breakdown of the Painlevé property does not necessarily imply that a PDE is non-integrable (many integrable non-Painlevé PDE are known to us, including some where there is, at present, no known change of variable to a Painlevé type equation). It is clear that equation (1.5) contains both DNLSI ($a = 2b$) and DNLSII ($b = 0$) as special cases. However, it is not a non-trivial generalisation since it may be transformed into both DNLSI and DNLSII when $a \neq b$ by a U(1) gauge transformation (Kundu 1984); the case $a = b$ transforms to a linear PDE. This shows that DNLSI and DNLSII are equivalent to each other, which apparently was first noticed by Wadati and Sogo (1983), although, as previously mentioned, the transformation is implied in the work of Kaup and Newell (1978) (see their equations (11) and (12)). Wadati and Sogo (1983) also found the Miura-type Bäcklund transformation connecting DNLSI with the original non-linear Schrödinger equation (NLS) (which was solved by inverse scattering by Zakharov and Shabat (1972)). Thus we conclude that condition (1.4) is at least sufficient for complete integrability and we conjecture that equation (1.1) is non-integrable otherwise because of the character of the singularities of $u(x, t)$ when (1.4) does not hold.

In a recent paper, Dodd and Fordy (1984) have used the prolongation method of finding the associated linear scattering problem due to Wahlquist and Estabrook (1975) in order to find conditions under which certain generalisations of the non-linear Schrödinger equation are (potentially) completely integrable. In their case II, which they narrow down essentially to our equation (1.1) with complex parameters, they find that the integrable members take the form (1.5), except that the last term appears with incorrect sign. However, they apparently did not recognise that their new integrable equation was transformable to DNLSI or, in the degenerate case $a = b$, to a linear equation.

In §§ 4 and 5 we consider some further non-linear Schrödinger-type equations. In particular we show that the mixed non-linear Schrödinger equation (MNLS) of Wadati *et al* (1979a):

$$\mathrm{i}u_t = u_{xx} + au^2u^* + \mathrm{i}b(u^2u^*)_x \tag{1.6}$$

where a and b are real constants, is equivalent to DNLSI under a point transformation, which was used by Kawada *et al* (1980) to simplify their boundary conditions on DNLSI. We give the results of a calculation analogous to that in § 3 which shows that the equation

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auu^*u_x + \mathrm{i}bu^2u_x^* + cu^3u^{*2} + du^2u^* \tag{1.7}$$

where a, b, c and d are real constants, denoted here the generalised mixed non-linear Schrödinger equation (GMNLS), has the Painlevé property only if (1.4) holds, regardless of the value of d; but when (1.4) holds, equation (1.7) is equivalent to (1.6) under the aforementioned gauge transformation. Also the aforementioned point transformation can be used to remove the last term from (1.7) whenever $a \neq b$. For convenience, the associated linear problem for (1.7) under the restriction (1.4), together with the effects of the gauge and point transformations, is written out at the end of § 4 so that the reader may readily adapt the inverse scattering formalism of Kaup and Newell (1978) or Kawata *et al* (1980) to any of the integrable equations in the GDNLS and GMNLS families.

In § 5, we investigate the singularities of the general solutions of the Wadati–Konno–Ichikawa equation (WKI) (Wadati *et al* 1979b)

$$-\mathrm{i}u_t = [u(1 + uu^*)^{-1/2}]_{xx} \tag{1.8}$$

which is known to be integrable and transformable to NLS, and the equation

$$\mathrm{i}u_t = u_{xx} + 2\mathrm{i}a(u^{3/2}u^{*1/2})_x + bu^2u^* \tag{1.9}$$

where a and b are real constants, which was discovered and conjectured to be integrable by Dodd and Fordy (1984). We find that equation (1.8) has movable square roots (movable fractional powers, especially square roots, are typical of equations transformable to Painlevé type equations via hodograph transformations) and that equation (1.9) has movable logarithms, which is highly suggestive that the equation is in fact *not* integrable.

Also in § 5 we simplify and construct the inverse of the Bäcklund transformation to the NLS equation, due independently to Ishimori (1982) and Wadati and Sogo

(1983). The inverse depends on several quadratures and the solution of a Riccati equation, which is precisely the same Riccati equation as in the Miura transformation relating NLS to DNLSI. It follows that WKI and DNLSI are related by a much simpler hodograph-type transformation which involves quadratures only in both directions. We point out the close analogy with the Bäcklund transformations relating the Harry-Dym equation (Kruskal 1975) to the KdV and MKdV equations. These results make the internal symmetries of the WKI equation more transparent and we derive a new Lie group of transformations, a two-to-one homomorphic image of SU(2), admitted by WKI.

2. Alternative formulations

Since we intend to carry out a Painlevé analysis on equation (1.1), we must first complexify all variables. In order to do this, write down both (1.1) and its formal complex conjugate ($\mathrm{i} \to -\mathrm{i}$, $u \to u^*$), and then replace u^* by v whenever it occurs. Thus we obtain the system

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auvu_x + \mathrm{i}bu^2v_x + cu^3v^2 \tag{2.1a}$$

$$-\mathrm{i}v_t = v_{xx} - \mathrm{i}auvv_x - \mathrm{i}bv^2u_x + cu^2v^3 \tag{2.1b}$$

in which u and v are treated as independent complex functions of the complex variables x and t (analogous to the situation for ODE (Ablowitz *et al* 1980a)).

In (2.1) make the change of variables

$$u(x, t) = R(x, t)\exp[\mathrm{i}\theta(x, t)] \qquad v(x, t) = R(x, t)\exp[-\mathrm{i}\theta(x, t)] \tag{2.2}$$

where R and θ are both to be regarded as complex functions. Then equations (2.1) become

$$R_t = 2R_x\theta_x + R\theta_{xx} + (a+b)R^2R_x \tag{2.3a}$$

$$-R\theta_t = R_{xx} - R\theta_x^2 + (b-a)R^3\theta_x + cR^5. \tag{2.3b}$$

Equation (2.3a) implies the existence of a potential ψ defined by

$$\psi_x = R^2 = uv \tag{2.4a}$$

$$\psi_t = 2R^2\theta_x + \tfrac{1}{2}(a+b)R^4 = \mathrm{i}(uv_x - vu_x) + \tfrac{1}{2}(a+b)u^2v^2. \tag{2.4b}$$

Suppose k is a real parameter, which will be chosen shortly; now define a new $\tilde{\theta}$ variable by

$$\tilde{\theta} = \theta + \mathrm{k}\psi \tag{2.5}$$

and also define

$$\tilde{u} = R\exp(\mathrm{i}\tilde{\theta}) = u\exp(\mathrm{i}k\psi) \qquad \tilde{v} = R\exp(-\mathrm{i}\tilde{\theta}) = v\exp(-\mathrm{i}k\psi). \tag{2.6}$$

This transformation is precisely the U(1) gauge transformation of Kundu (1984), which also applies essentially unchanged to the more general equation (1.7), with no new restrictions on the parameters. The new variables $(\tilde{u}, \tilde{v})$ and $(R, \tilde{\theta})$ satisfy equations identical in form to equations (2.1) and (2.3), respectively, with new parameters

$$\tilde{a} = a - 2k \qquad \tilde{b} = b - 2k \qquad \tilde{c} = c + k^2 + \tfrac{1}{2}k(a - 3b). \tag{2.7}$$

It follows that at most two of the three parameters a, b and c are essential and that various normalisations are possible, e.g. $\tilde{c}=0$ (in fact equations (2.1) also possess a scaling symmetry, which is obvious by inspection, and therefore actually only one of the three parameters is essential). The most convenient normalisation, from the point of view of Painlevé analysis, is $\tilde{a}+\tilde{b}=0$. So we choose

$$k=\tfrac{1}{4}(a+b) \qquad T=\theta+\tfrac{1}{4}(a+b)\psi. \tag{2.8}$$

Therefore by eliminating θ in favour of T in equation (2.3) we obtain

$$R_t=RT_{xx}+2R_xT_x \tag{2.9a}$$

$$-RT_t=R_{xx}-RT_x^2+AR^3T_x+BR^5 \tag{2.9b}$$

where

$$A:=b-a \qquad B:=c+\tfrac{1}{16}(a+b)(3a-5b). \tag{2.9c}$$

In terms of R and T, the relations defining the potential ψ are

$$\psi_x=R^2 \qquad \psi_t=2R^2T_x. \tag{2.10}$$

We conclude this section by writing down the relationships between DNLSI, DNLSII and NLS (due to Wadati and Sogo (1983)) in the present notation. Let u_1 solve DNLSI in the form (1.2) and write $v_i=u_i^*$, $i=1,2,3$ (u_3, v_3 will be required shortly). Then a solution of DNLSII (1.3) is given by

$$u_2=(b/a)^{1/2}u_1\exp(\mathrm{i}b\psi/2) \qquad v_2=(b/a)^{1/2}v_1\exp(-\mathrm{i}b\psi/2) \tag{2.11}$$

where

$$\psi_x=u_1v_1=(a/b)u_2v_2 \tag{2.12a}$$

$$\begin{aligned}\psi_t&=\mathrm{i}(u_1v_{1x}-v_1u_{1x})+\tfrac{3}{2}bu_1^2v_1^2\\&=\mathrm{i}(a/b)(u_2v_{2x}-v_2u_{2x})+\tfrac{1}{2}(a^2/b)u_2^2v_2^2\end{aligned} \tag{2.12b}$$

(a scaling of u_2 and v_2 has been used to match the parameters in (1.2) and (1.3)). Before considering the Miura transformation to NLS, first apply the gauge transformation (2.6) with $k=b$ to DNLSI to obtain the following equation (Gerdjikov and Ivanov 1982), which may conveniently be denoted DNLSIII

$$\mathrm{i}u_t=u_{xx}-\mathrm{i}bu^2v_x+\tfrac{1}{2}b^2u^3v^2 \tag{2.13a}$$

$$-\mathrm{i}v_t=v_{xx}+\mathrm{i}bv^2u_x+\tfrac{1}{2}b^2u^2v^3 \tag{2.13b}$$

which is solved by

$$u_3=u_1\,\mathrm{e}^{\mathrm{i}b\psi} \qquad v_3=v_1\,\mathrm{e}^{-\mathrm{i}b\psi}. \tag{2.14}$$

Now the original NLS equation, namely

$$\mathrm{i}u_t=u_{xx}+bu^2v \tag{2.15a}$$

$$-\mathrm{i}v_t=v_{xx}+buv^2 \tag{2.15b}$$

is solved by

$$u=u_3 \qquad v=-\mathrm{i}v_{3x}+\tfrac{1}{2}bu_3v_3^2 \tag{2.16a}$$

and also by

$$u=\mathrm{i}u_{3x}+\tfrac{1}{2}bu_3^2v_3 \qquad v=v_3. \tag{2.16b}$$

The two distinct Miura transformations (2.16a, b) lead immediately to auto-Bäcklund transformations for all integrable members of the GDNLS and GMNLS families.

3. Painlevé analysis

In this section we use the method given by Weiss *et al* (1983), simplified by Kruskal (1982) (hereafter called the WTCK method), in order to determine necessary conditions for equations (2.9) to possess the Painlevé property for PDE and hence be possibly completely integrable (equivalently we could have considered either equations (2.1) or (2.3)). Painlevé analysis strongly suggests, but does not prove, that these conditions are *necessary* for the equations (2.9) to be completely integrable since the solutions have a rather bad type of movable singularity when the conditions are not met. In order to be more precise about the necessity of the condition, we would need a definition of integrability, which we are not ready to give; whatever definition is eventually agreed upon, we believe the set of Painlevé-type PDE should be a proper subset of the set of integrable PDE. On the other hand, the conditions turn out to be sufficient for integrability because, when they are met, the equation is transformable to DNLSI, whose solution by inverse scattering has been provided by Kaup and Newell (1978). Furthermore, when the same conditions are met, we claim that the PDE has the Painlevé property (except for an overall multiplicative factor which can be removed by a change of variables) on the basis that all solutions derived from the inverse scattering formalism have no movable singularities worse then poles (see Ablowitz *et al* (1980a, b) and McLeod and Olver (1983) for the general argument); a complete proof would require showing that the Painlevé property is not influenced by the class of solutions not captured by the inverse scattering formalism due to restrictions on the initial value problem.

The WTCK method involves seeking solutions of equations (2.9) in the form

$$R(x,t)=X^{p}\sum_{j=0}^{\infty}R_j(t)X^j \qquad T(x,t)=X^{q}\sum_{j=0}^{\infty}T_j(t)X^j \tag{3.1}$$

with

$$X(x,t):=x-f(t) \tag{3.2}$$

where $f(t)$ is an arbitrary locally analytic function of t and $R_j(t)$ and $T_j(t)$, $j=0,1,2,\ldots$, are locally analytic functions of t, with R_0 and T_0 not both zero, valid in the neighbourhood of a generic non-characteristic hypersurface (in this case curve) defined by $X=0$. It should be remarked that we attempt to construct expansions of the form (3.1) even though we expect them to break down for some j or even be of the wrong form altogether; the justification for the WTCK method is provided by the fact that the postulated forms are correct when the conditions for the Painlevé property are satisfied, and in other cases the Painlevé analysis shows precisely what forms the correct expansions should take (of course, a more satisfying but far less convenient procedure would be to write down a formal expansion sufficiently general to cover all anticipated cases, for example a logarithmic psi series, and then derive the recursion relations for the coefficients and in particular show that extraneous terms have zero coefficients). Another comment we would like to include concerns the nature of 'characteristic' hypersurfaces. The special choice (3.2) excludes the cases $X(x,t)=f(t)=t-t_0$ (without loss of generality); the lines $t-t_0=0$ are characteristics for the PDE (2.9) in the context of the usual Cauchy problem, where we prescribe arbitrary Cauchy data on a non-singular non-characteristic hypersurface $X(x,t)=0$. Of course, since Painlevé tests are concerned with generic singular hypersurfaces, we are free to exclude special hypersurfaces for reasons of convenience alone, and it may be necessary to exclude some because the proposed series expansion breaks down. However, it is very important to identify and exclude characteristic hypersurfaces (especially when the PDE are not semilinear) because singularities of arbitrary character can live on characteristics and such singularities, no matter how 'movable' they appear to be, have no bearing of the Painlevé property of the PDE.

Substituting (3.1) and (3.2) into (2.9) and equating coefficients of like powers of X determines p and q and defines recursion relations for the R_j, T_j, $j=0,1,\ldots$. Essentially, it is required that the recursion relations be consistent so that the series (3.1) contain the requisite number of arbitrary functions as required by the Cauchy-Kowalevski theorem (in this case four) in order that (3.1) represent the general integral. It is usually also required that p and q be integers; however, we drop this requirement since any non-Painlevé behaviour at leading order (movable fractional or irrational powers or, if an exponential factor were present, movable essential singularities) could easily be removed by an obvious transformation.

In order to determine the leading order behaviour of R and T in the neighbourhood of the singularity manifold defined by $X(x,t)=0$, we assume that

$$R = R_0X^p + \mathrm{o}(X^p) \qquad T = T_0X^q + \mathrm{o}(X^q) \tag{3.3}$$

and that the 'little oh' terms do not take the lead after differentiation. Substituting (3.3) into (2.9) and balancing powers of X shows that the dominant terms in (2.9*b*) vary according as to whether the constants A and B are non-zero or zero. There are three cases to consider:

(i) $B \neq 0$, A arbitrary,
(ii) $B=0$, $A\neq 0$,
(ii) $A=B=0$.

Case 1. $B\neq 0$, A arbitrary (i.e.,, $16c+(a+b)(3a-5b)\neq 0$). In this case, in the neighbourhood of the singularity manifold defined by $X(x,t)=0$, using the WTCK method, we obtain the following formal logarithmic psi-series expansions for R and T:

$$\begin{aligned} R(x,t) = {} & R_0X^{-1/2} + R_1X^{1/2} + R_2X^{3/2} + (R_3 + R_{3,1}\ln X)X^{5/2} + (R_4 + R_{4,1}\ln X)X^{7/2} \\ & + (R_5 + R_{5,1}\ln X)X^{9/2} + (R_6 + R_{6,1}\ln X + R_{6,2}(\ln X)^2)X^{11/2} \\ & + \mathrm{O}(X^{13/2}(\ln X)^2) \end{aligned} \tag{3.4a}$$

$$\begin{aligned} T(x,t) = {} & T_0 + T_1X + T_2X^2 + T_3X^3 + T_4X^4 + (T_5 + T_{5,1}\ln X)X^5 \\ & + (T_6 + T_{6,1}\ln X)X^6 + \mathrm{O}(X^7\ln X) \end{aligned} \tag{3.4b}$$

where the R_j, $R_{j,k}$, T_j, $T_{j,k}$ are (locally analytic) functions of t. The resonances are given by

$$(j+1)j(j-2)(j-3)=0 \qquad \text{valid for } j\geqslant 1. \tag{3.5}$$

The resonances are the values of j at which arbitrary functions arise in the expansions (3.4) due to the vanishing of the coefficient determinant of the pair of recursion relations defining $R_{j,k}$ and $T_{j,k}$, and for every non-negative integer resonance there is a compatibility condition which must be identically satisfied in order that ((2.9) has a general integral of the form (3.1) (i.e. no logarithms); if a positive non-integer resonance occurred, that would normally indicate that fractional powers of X should be included in (3.1), and so on. The four arbitrary functions of integration in the expansions (3.4a, b) are

$$f(t) \qquad T_0(t) \qquad T_2(t) \qquad R_3(t). \tag{3.6}$$

The resonance $j=-1$ is usually associated with $f(t)$ being arbitrary (cf Weiss *et al* (1983)—see comment in case 2 below), i.e. we are expanding about an arbitrary non-characteristic hypersurface. The resonance $j=0$ is checked separately and arises from the fact that there is only one equation at leading order (order $X^{-5/2}$) and it only determines $R_0(t)$, leaving $T_0(t)$ arbitrary. The first few coefficients in the expansions (3.4a, b) are

$$R_0=(-3/4B)^{1/4} \text{ (four values)} \qquad R_1(t)=-\tfrac{1}{8}A(R_0)^3f'(t)$$
$$T_0(t) \text{ arbitrary} \qquad T_1(t)=-\tfrac{1}{2}f'(t)$$

where $' := \mathrm{d}/\mathrm{d}t$. We find that the compatibility condition corresponding to the resonance $j=2$ is identically satisfied and so there no logarithmic terms need to be introduced. However, since the compatibility condition corresponding to the resonance $j=3$ is not satisfied in general, it is necessary to introduce a logarithmic term into the expansion for R with coefficient

$$R_{3,1}(t)=\tfrac{1}{96}(3A^2-16B)(R_0)^5f''(t). \tag{3.7}$$

Some coefficients of higher logarithmic terms are

$$\begin{aligned}&R_{4,1}=-\tfrac{3}{40}A(R_0)^2R_{3,1}f' \qquad T_{5,1}=(R'_{3,1}-12T_2R_{3,1})/(15R_0)\\&R_{6,2}=\tfrac{5}{14}(R_{3,1})^2/R_0.\end{aligned} \tag{3.8}$$

From (3.7) it follows that a necessary condition for the absence of movable logarithmic terms in the expansions is

$$3A^2-16B=0 \qquad \text{i.e. } c=\tfrac{1}{4}b(2b-a). \tag{3.9}$$

Thus in the case $3A^2-16B=0$, $B\neq 0$, the variables R^2 and T have no movable critical hypersurfaces (curves) of the special form implied by the logarithmic terms (3.7) and (3.8). We emphasise once again that we have not proved that equations (2.9) have the Painlevé property in the case $3A^2-16B=0$; the Painlevé property is a global property that cannot be proved in any reasonably practical way by looking at local expansions such as (3.1). In this case a partial proof of the Painlevé property may be given *a posterori* by using the fact that we know how to solve equations (2.9) by the inverse scattering method when $3A^2-16B=0$.

Now the relation (3.9) is precisely the condition that GDNLS be equivalent to DNLSI and DNLSII under the gauge transformation (2.6) (provided $a\neq b$; the case $a=b$ belongs to case 3 below). If u_1 and $v_1(=u_1^*)$ solve DNLSI (1.2) and u_2 and $v_2(=u_2^*)$ solve DNLSII (1.3), then a solution of GDNLS, which now has the special form (1.5) with $a\neq b$, is given by

$$u=\left(\frac{b}{a-b}\right)^{1/2}u_1\exp(\mathrm{i}k_1\psi)=\left(\frac{a}{a-b}\right)^{1/2}u_1\exp(\mathrm{i}k_2\psi) \tag{3.10a}$$

$$v=\left(\frac{b}{a-b}\right)^{1/2}v_1\exp(-\mathrm{i}k_1\psi)=\left(\frac{a}{a-b}\right)^{1/2}v_2\exp(-\mathrm{i}k_2\psi) \tag{3.10b}$$

where ψ is defined by equations (2.12a, b) and

$$k_1=\frac{b(a-2b)}{2(a-b)} \qquad k_2=-\frac{b^2}{2(a-b)}. \tag{3.11}$$

We conclude the discussion of this case with some remarks about similarity reductions of equations (2.9). Consider the scaling reduction

$$R(x, t) = t^{-1/4}R(z) \qquad T(x, t) = T(z) + h \ln t \qquad z = xt^{-1/2} \tag{3.12}$$

where h is a complex constant. Then $R(z)$ and $T(z)$ satisfy

$$-\tfrac{1}{4}R - \tfrac{1}{2}zR' = RT'' + 2R'T' \tag{3.13a}$$

$$R(\tfrac{1}{2}zT' - h) = R'' - R(T')^2 + AR^3T' + BR^5 \tag{3.13b}$$

where $' := \mathrm{d}/\mathrm{d}z$. Equation (3.13$a$) can be integrated once after multiplication by R, which gives T' in terms of R^2. Substituting for T' in (3.13b) and defining $w := R^2$ gives a second-order ODE for w which is of Painlevé type if and only if

$$3A^2 - 16B = 0 \qquad \text{i.e. } c = \tfrac{1}{4}b(2b - a). \tag{3.14}$$

Unless (3.14) holds, logarithmic terms arise in the series expansion of w (and hence also of R) in the neighbourhood of a 'pole' at z_0 at the $w_{3,1}$ term (corresponding to the $R_{3,1}$ term in (3.4a)). In fact, the series expansions for $R(z)$ and $T(z)$ about $z = z_0$ have a very similar structure to the series expansion (3.4a, b),except that the coefficients are constants instead of functions of t. Note also that if (3.14) holds then $w(z)$ satisfies the fourth Painlevé equation (Ince 1956)

$$w'' = (1/2w)(w')^2 + \tfrac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \beta/w \tag{3.15}$$

where α and β are constants, after appropriately rescaling w and z.

However, it we consider the travelling wave solution of (2.9)

$$R(x, t) = R(z) \qquad T(x, t) = T(z) + \mu t \qquad z = x - \lambda t \tag{3.16}$$

where μ and λ are constants, then R and T are solvable in terms of Jacobian elliptic functions, so that R^2 and T' are meromorphic throughout the complex z plane. In the series expansions of R and T, there are *no* logarithmic terms corresponding to those that arise in (3.4a, b) for any A or B, since for the travelling wave reduction, the singularity manifold satisfies $f''(t) = 0$ (recall equation (3.7)).

Case 2. $B = 0$ and $A \neq 0$ (i.e. $16c + (a+b)(3a - 5b) = 0$ and $a \neq b$). In this case, in the neighbourhood of the singularity manifold defined by (3.2), using the WTCK method, we obtain the following formal logarithmic psi-series expansions for the R and T:

$$\begin{aligned} R(x, t) = {} & R_1X^{-1} + R_2 + (R_3 + R_{3,1} \ln X)X + (R_4 + R_{4,1} \ln X)X^2 \\ & + (R_5 + R_{5,1} \ln X + R_{5,2}(\ln X)^2)X^3 + \mathrm{O}(X^4(\ln X)^2) \end{aligned} \tag{3.17a}$$

$$\begin{aligned} T(x, t) = {} & T_0 + T_1X + T_2X^2 + (T_3 + T_{3,1} \ln X)X^3 + (T_4 + T_{4,1} \ln X)X^4 \\ & + (T_5 + T_{5,1} \ln X + T_{5,2}(\ln X)^2)X^5 + \mathrm{O}(X^6(\ln X)^2) \end{aligned} \tag{3.17b}$$

where the R_j, $R_{j,k}$, T_j, $T_{j,k}$ are (analytic) functions of t. It may seem unusual to label the leading coefficient in the expansion (3.17a) R_1. However, if it is labelled R_0, then the recursion relations take the form of a pair of linear equations for $R_{j-1,k}$ and $T_{j,k}$. Therefore it seems more natural to write the expansion as above and take the view that the R_0X^{-2} term is there but has a vanishing coefficient. The resonances are given by

$$j^2(j-3)(j-5) = 0 \qquad \text{valid for } j \geq 2 \tag{3.18}$$

and after checking $j=0$ separately, we find resonances $j=0,3,5$. The four arbitrary functions in the expansions (3.17a, b) are

$$f(t) \quad T_0(t) \quad T_3(t) \quad R_5(t).$$

The first few terms are

$$R_1(t)=\pm(4/Af'(t))^{1/2} \qquad R_2(t)=\tfrac{1}{3}R_1f''/(f')^2$$
$$T_1(t)=-\tfrac{1}{2}f'(t) \qquad T_2(t)=\tfrac{1}{4}f''(t)/f'(t).$$

The compatibility condition corresponding to the resonance $j=3$ is not identically satisfied and it is necessary to introduce logarithmic terms in the expansion. The first logarithmic terms have coefficients

$$T_{3,1}=\tfrac{1}{9}(f')^{-3}\{f'f'''-3(f'')^2\} \qquad R_{3,1}=(2/f')R_1T_{3,1}. \tag{3.19}$$

Some higher coefficients of logarithmic terms are

$$T_{5,2}=-\tfrac{12}{5}f'(T_{3,1})^2 \qquad R_{5,3}=0. \tag{3.20}$$

Note that the resonance $j=5$ would be expected to create a $R_{5,3}X^3(\ln X)^3$ term because of the $R_{5,1}$ and $R_{5,2}$ terms which already arise due to the coupling with the product of the $R_{3,1}$ and $T_{3,1}$ terms; however, a cancellation occurs with gives $R_{5,3}=0$. Equation (3.19) shows that movable logarithms must always arise in the expansions for case 2, and therefore Painlevé analysis strongly suggests that, in this case, equation (2.1) is non-integrable.

If we consider the similarity reductions (3.12) and (3.16), then logarithmic terms corresponding to the $R_{3,1}$ and $T_{3,1}$ terms in (3.1a, b) do *not* arise in the series expansions of $R(z)$ and $T(z)$ in the neighbourhood of a 'pole' at z_0, since for both reductions $f'f'''-3(f'')^2=0$ (recall equation (3.19)). However for the scaling reduction (3.12), logarithmic terms do appear at the $j=5$ resonance and we conclude that the scaling reduction cannot have the Painlevé property in case 2. Notice that the scaling and travelling wave reductions have qualitatively different singularity structures to the full PDE in case 2.

The absence of the factor $j+1$ on the left-hand side of equation (3.18) deserves a brief comment. It is reasonably easy to show directly that, for a generic system of ODE or PDE is of total order k, the set of recursion relations which express the jth coefficient in an expansion analogous to equation (3.1) (counting the leading coefficients a 'zeroth') have coefficient determinant in the form of a polynomial in j of degree k, one of whose roots is $j=-1$. This can never be a valid resonance, of course, since the terms of the series begin at $j=0$ and the derivation of the recursion relations is valid only for $j\geqslant j_0$, for some $j_0\geqslant 1$, the first few coefficients requiring a separate treatment. Some authors remark that the false resonance $j=-1$ corresponds to the arbitrary function which describes the singular hypersurface (such as $f(t)$ in equation (3.2)). Observing the factor $j+1$ provides a useful check on the calculations and so it is worthwhile to be aware of possible exceptions, of which there are several types. A sufficient condition which will guarantee that $j+1$ is a factor is that the leading terms in the expansions in ascending powers of X of the differential equations themselves (after the formal series solutions have been substituted in) involve only the $j=0$ terms. This condition can be violated if one of the indices $p, q, \ldots,$ as in equation (3.1), is zero or a small positive integer, as is the case in equation (3.1b) where $q=0$. For example, the general solution of the ODE

$$y'y'''-2(y'')^2+18y'=xy(y')^2 \tag{3.21}$$

where $' := \mathrm{d}/\mathrm{d}x$, in the neighbourhood of a logarithmic branch point at x_0, has an expansion

$$y = a_0 + a_3(x-x_0)^3 + a_4(x-x_0)^4 + \ldots + a_8(x-x_0)^8 + (x-x_0)^9[a_9 + a_{9,1}\ln(x-x_0)] + \ldots \tag{3.22}$$

in which the arbitrary constants of integration are x_0, a_0 and a_9 and the recursion relation takes the form

$$j(j-2)(j-9)a_j = F(a_0, a_3, \ldots, a_{j-1}; x_0) \qquad j \geqslant 4. \tag{3.23}$$

In this case the $j=3$ terms are dominant on the left-hand side of (3.21) and the false resonance occurs at $j=2$. A different type of exception to the $j=-1$ rule is provided by the PDE

$$u_{xx}u_{yy} - (u_{xy})^2 = 0 \tag{3.24}$$

whose general integral can be expanded in the form

$$u = u_0(x) + u_1(x)X + u_{3/2}(x)X^{3/2} + u_2(x)X^2 + u_{5/2}(x)X^{5/2} + \ldots \tag{3.25}$$

where $X := y - f(x)$. In this case the arbitrary functions are $f(x)$ and $u_0(x)$ and the recursion relation takes the form

$$j(j-\tfrac{1}{2})u_j(x) = F(u_0, u_1, u_{3/2}, \ldots, u_{j-1/2}; x, f(x)) \qquad j \geqslant 2 \tag{3.26}$$

where F depends on the indicated arguments and their first two derivatives. Note that the $j=\frac{3}{2}$ terms are dominant in the second derivatives of u and so it is perhaps not surprising that the false resonance occurs at $j=\frac{1}{2}$. There are many other possible exceptional series expansions of qualitatively different character, including some where the false resonance is absent altogether.

Case 3. $A = B = 0$ (i.e., $a = b$, $4c = b^2$). In this case, equations (2.1) have the Painlevé property (as do the variables R^2 and T_x) and can be solved as follows. We define new variables

$$\tilde{u} = R\exp(\mathrm{i}T) = u\exp(\tfrac{1}{2}\mathrm{i}b\psi) \qquad \tilde{v} = R\exp(-\mathrm{i}T) = v\exp(-\tfrac{1}{2}\mathrm{i}b\psi). \tag{3.27}$$

Then $\tilde{u}$ and $\tilde{v}$ satisfy the linear uncoupled PDE

$$\mathrm{i}\tilde{u}_t = \tilde{u}_{xx} \qquad -\mathrm{i}\tilde{v}_t = \tilde{v}_{xx}. \tag{3.28}$$

Linear PDE such as (3.28) are easily solvable by integral transform methods.

Therefore in all three cases, Painlevé analysis strongly suggests that a necessary condition for the system of PDE (2.1) to be completely integrable is

$$c = \tfrac{1}{4}b(2b-a). \tag{3.29}$$

Furthermore, this condition is a sufficient condition for integrability since, when satisfied, it is then possible to make a change of variables in order to transform (2.1) into either DNLSI or the linear equation (3.28). If (3.29) holds, then equations (2.1) become

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auvu_x + \mathrm{i}bu^2v_x + \tfrac{1}{4}b(2b-a)u^3v^2 \tag{3.30a}$$

$$-\mathrm{i}v_t = v_{xx} - \mathrm{i}auvv_x - \mathrm{i}bv^2u_x + \tfrac{1}{4}b(2b-a)u^2v^3. \tag{3.30b}$$

The Painlevé-type expansions for the original variables u and v now take the forms

$$u = X^{-p} \sum_{j=0}^{\infty} u_x X^j \qquad v = X^{p-1} \sum_{j=0}^{\infty} v_j X^j \tag{3.31}$$

with $p = a/(a-b)$, and X as in (3.2), provided that $a \neq b$ (if $a = b$ then equations (3.30) can be transformed into an uncoupled system of linear PDE—case 3 above). The resonances are given by

$$(j+1)j(j-2)(j-3) = 0 \qquad \text{valid for } j \geq 1.$$

The compatibility conditions associated with the resonances $j = 2$ and $j = 3$ are identically satisfied and the four arbitrary functions are

$$f(t) \quad \text{either } u_j(t) \text{ or } v_j(t) \quad j = 0, 2, 3.$$

Therefore equations (3.30) have solutions of the form (3.31) with the requisite number of arbitrary functions.

Note that the only restriction upon a or b is $a \neq b$, and so p may take *any* real value, rational or irrational. Therefore unless p is an integer, the variables u and v exhibit non-Painlevé behaviour at leading order (movable fractional or irrational powers according to whether p is a rational fraction or irrational). However, this non-Painlevé behaviour is trivially removable, for example, by transforming to the R and T variables and therefore has no bearing on the question of integrability. Additionally, note that the leading order powers of u and v are *not* equal, in general, since we are not, for the purposes of Painlevé analysis, restricting attention to the original situation where u and v are mutually complex conjugate functions of real variables.

4. The generalised mixed non-linear Schrödinger equation

The techniques of § 3 apply essentially unchanged to the generalised mixed non-linear Schrödinger equation (GMNLS)

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}auvu_x + \mathrm{i}bu^2v_x + cu^3v^2 + du^2v \tag{4.1a}$$

$$\mathrm{i}v_t = v_{xx} - \mathrm{i}auvv_x - \mathrm{i}bv^2u_x + cu^2v^3 + duv^2 \tag{4.1b}$$

(a, b, c, d real constants), which generalises both GDNLS (2.1) and the mixed non-linear Schrödinger equation (MNLS) of Wadati *et al* (1979a)

$$\mathrm{i}u_t = u_{xx} + du^2v + \mathrm{i}b(u^2v)_x \tag{4.2a}$$

$$-\mathrm{i}v_t = v_{xx} + duv^2 - \mathrm{i}b(uv^2)_x. \tag{4.2b}$$

Equation (4.2) is equivalent to DNLSI under the point transformation

$$\tilde{u} = u \exp[-\mathrm{i}(\alpha x + \alpha^2 t)] \qquad \tilde{v} = v \exp[\mathrm{i}(\alpha x + \alpha^2 t)] \tag{4.3a}$$

$$\tilde{x} = x + 2\alpha t \qquad \tilde{t} = t \tag{4.3b}$$

with the choice $\alpha = d/b$. The variables $\tilde{u}(\tilde{x}, \tilde{t})$ and $\tilde{v}(\tilde{x}, \tilde{t})$ satisfy equations (4.2) with tildes on all variables, except that $\tilde{d} = 0$, $\tilde{b} = b$. This transformation was used by Kawata *et al* (1980) to solve the initial value problem for DNLSI by inverse scattering under the restriction $u(x, t) \sim$ (constant) $\exp[\mathrm{i}(kx - \omega t)]$ as $x \to \pm\infty$, which was more general than the earlier treatments of Kaup and Newell (1978) and Kawata and Inoue (1978).

With the choice

$$\alpha = d/(a-b) \qquad a \neq b \tag{4.4}$$

the same point transformation (4.3) converts GMNLS (4.1) to GDNLS (2.1) with tildes on all variables except the parameters a, b and c, which are unaffected. If α is allowed to be arbitrary, then (4.1) transforms to an equation of the same form with

$$\tilde{a} = a \qquad \tilde{b} = b \qquad \tilde{c} = c \qquad \tilde{d} = d - \alpha(a-b) \tag{4.5}$$

which shows incidentally that (4.3) has no effect when $a = b$. In addition to (4.3), equation (4.1) also admits the gauge transformation (2.6), where the potential ψ is defined by (2.4) regardless of the value of d. The parameters a, b and c transform according to (2.7), and d is unaffected.

This discussion limits the number of new cases which need to be Painlevé tested. Define new parameters A and B by (2.9c) above; then since the du^2v and duv^2 terms are never dominant (except in the case $a = b = c = 0$, which is just NLS), the Painlevé testing splits into the same three cases as in § 3. In case 3, where $A = B = 0$, $a = b$, $4c = b^2$, GMNLS is equivalent to NLS under the gauge transformation, and so is integrable. In case 2, where $A \neq 0$, $B = 0$, we have that $a \neq b$ and so GMNLS is equivalent to GDNLS under the point transformation. In § 3, we found that this case fails the Painlevé test and so is probably not integrable. In case 1, where $B \neq 0$ and A is unrestricted, we know that GMNLS is equivalent to GDNLS under the point transformation. In § 3, we found that this case fails the Painlevè test and so is probably not integrable. In case 1, where $B \neq 0$ and A is unrestricted, we know that GMNLS is equivalent to GDNLS whenever $A \neq 0$ (recall $A = b - a$). In that case, the necessary condition that (4.1) has the Painlevé property except for the overall multiplicative factors X^{-p} and X^{p-1} as in equation (3.31) (which are only an artefact of the choice of variables) is

$$c = \tfrac{1}{4}b(2b-a) \tag{4.6}$$

which is also satisfied identically in case 3.

The only remaining case to be tested is case 1 with $a = b$ and $4c \neq b^2$. Under the gauge transformation (2.6), this case can be put in the canonical form

$$\mathrm{i}u_t = u_{xx} + du^2v + cu^3v^2 \tag{4.7a}$$

$$-\mathrm{i}v_t = v_{xx} + duv^2 + cu^2v^3 \tag{4.7b}$$

with $c \neq 0$. In terms of the polar variables R and T, where $u = R\,\mathrm{e}^{\mathrm{i}T}$, $v = R\,\mathrm{e}^{-\mathrm{i}T}$, we find logarithmic psi-series expansions for R and T of precisely the same forms as (3.4a, b). The arbitrary functions are $f(t)$, $T_0(t)$, $T_2(t)$ and $R_3(t)$. The first term involving a movable logarithm is $R_{3,1}X^{5/2}\ln X$, where $X := x - f(t)$ and

$$R_{3,1} = \tfrac{1}{8}R_0 f''(t) \qquad R_0^4 = -3/4c. \tag{4.8}$$

Since $R_{3,1}$ is independent of d, it is not possible to choose d so as to make $R_{3,1}$ vanish identically for generic $f(t)$. We conclude that equation (4.7) always has movable logarithms when $c \neq 0$, and so is probably not integrable. (Of course, just as in § 3, the travelling wave reduction is free of logarithms and integrable in terms of elliptic functions.)

Thus all cases of GMNLS free of movable logarithms satisfy (4.6) and we conjecture that these are the only integrable cases. Further evidence against integrability when (4.6) does not hold is that the equations then do not admit a non-trivial Wahlquist-Estabrook prolongation structure (Dodd and Fordy 1984). However, one must be careful when judging integrability on the basis of internal symmetries; for example, Painlevé's first equation

$$d^2y/dx^2 = 6y^2 + x \tag{4.9}$$

admits only a finite group of discrete scalings of order 5, but any reasonable definition of integrability should certainly include this equation. When (4.6) holds, GMNLS is equivalent under the gauge transformation (2.6) to either MNLS (4.2) (case 1) or NLS (2.15) (case 3), and, in the former case, the point transformation (4.3) establishes the equivalence with DNLSI (1.2).

A more general starting point would be to let all twelve terms in equations (4.1*a*, *b*) have independent complex coefficients (the coefficients of u_{xx} and v_{xx} can be normalised to unity). In that case, two different sets of conditions must be satisfied in order that u and v admit expansions of the form (3.31). The first can be found by substituting (3.31) into the PDE and requiring that the resonances be distinct integers, positive or negative, but the justification for the procedure (especially the interpretation of negative fractional resonances) requires using an argument similar to that in § 14.31 of Ince (1956, p 327), which is based on Painlevé's α-test. Generically, eight algebraic expressions involving the coefficients must take integer values, two for each value of u_0v_0, which itself satisfies a quartic equation. We have not fully explored all the various possibilities, but we are reasonably confident that the only way to satisfy this first set of conditions is to let the coefficients of u_t, v_t, u^2v and uv^2 be arbitrary while the remaining terms have coefficients restricted to the same forms as in (4.1). In that case, the resonances occur at $j = 0, 2, 3$, and then the second set of conditions, namely freedom from logarithms, requires that the coefficients be exactly as in (4.1) (perhaps after rescaling) (from $j = 2$ resonance) and satisfy (4.6) (from $j = 3$ resonance).

For convenience, we write down the associated linear problem for GMNLS subject to the integrability condition (4.6). It will contain two spectral parameters λ and μ related by

$$\lambda^2 = (a-b)\mu - \tfrac{1}{2}d. \tag{4.10}$$

It is natural to eliminate μ in favour of λ whenever $a \neq b$.

$$w_{1,x} = -\mathrm{i}[\mu + \tfrac{1}{4}(a-2b)uv]w_1 + \lambda v w_2 \tag{4.11a}$$

$$w_{2x} = \lambda u w_1 + \mathrm{i}[\mu + \tfrac{1}{4}(a-2b)uv]w_2 \tag{4.11b}$$

$$w_{1,t} = Aw_1 + Bw_2 \tag{4.12a}$$

$$w_{2,t} = Cw_1 - Aw_2 \tag{4.12b}$$

where

$$A = -2\mathrm{i}\mu^2 - \mathrm{i}\lambda^2 uv - \tfrac{1}{4}\mathrm{i}(a-2b)[\mathrm{i}(uv_x - vu_x) + \tfrac{1}{2}(a+b)u^2v^2] \tag{4.13a}$$

$$B = \lambda(2\mu v + \mathrm{i}v_x + \tfrac{1}{2}auv) \tag{4.13b}$$

$$C = \lambda(2\mu u - \mathrm{i}u_x + \tfrac{1}{2}auv). \tag{4.13c}$$

The x-derivative part (4.11) is a simple generalisation of the standard AKNS (Ablowitz *et al* 1974) and Kaup–Newell (Kaup and Newell 1978) spectral problems and may be transformed into either the standard forms by the gauge and point transformations already discussed (Wadati and Sogo 1983, Kundu 1984). The effect of the combined gauge and point transformations on the variables and parameters is

$$u' = u\,e^{i\Psi} \qquad v' = v\,e^{-i\Psi} \tag{4.14a}$$

$$w_1' = w_1\,e^{-i\Psi/2} \qquad w_2' = w_2\,e^{i\Psi/2} \tag{4.14b}$$

$$x' = x + 2\alpha t \qquad t' = t \tag{4.14c}$$

$$\psi' = \psi \tag{4.14d}$$

$$a' = a - 2k \qquad b' = b - 2k \qquad d' = d - \alpha(a-b) \tag{4.14e}$$

$$\lambda' = \lambda \qquad \mu' = \mu - \tfrac{1}{2}\alpha \tag{4.14f}$$

where

$$\Psi := k\psi - \alpha x - \alpha^2 t$$

and the potential ψ is defined by (2.4) (valid for $d \neq 0$). It is worth noting that the initial values $u'(x', 0)$ and $v'(x', 0)$ of the primed variables are easily reconstructed from the initial values of the original variables and that the Kaup–Newell boundary condition, $u(x, 0) \to 0$, $v(x, 0) \to 0$ as $x \to \pm\infty$ (sufficiently fast so that $\psi(x, 0)$ is bounded), is preserved by the primed variables. (See Kawata *et al* (1980) for the treatment of the more general boundary condition, $u(x, t) \sim$ (constant) $\exp[i(\kappa x - \omega t)]$ as $x \to \pm\infty$.)

5. Other non-linear Schrödinger equations

5.1. The Dodd–Fordy equation

In their investigation of the integrability of certain generalisations of the NLS equation, Dodd and Fordy (1984) presented the equation

$$iu_t = u_{xx} + 2ia(u^{3/2}v^{1/2})_x + bu^2v \tag{5.1a}$$

$$-iv_t = v_{xx} - 2ia(u^{1/2}v^{3/2})_x + buv^2 \tag{5.1b}$$

as a candidate for complete integrability. Painlevé analysis of this equation suggests that it is probably not integrable since it has movable logarithms of the sort encountered in § 3, although this argument is far from watertight. Dodd and Fordy's argument in favour of integrability is that equation (5.1) has a non-trivial prolongation structure, which leaves open the possibility that it may have some interesting exact solutions and that it has an associated linear problem. However, the latter cannot be used as a starting point for inverse scattering since the spectral parameter can be removed by a trivial gauge transformation.

In order to run the Painlevé test, complexify all variables in (5.1) and convert to polar form:

$$u = R\,e^{iT} \qquad v = R\,e^{-iT}. \tag{5.2}$$

The analysis splits into two cases: case 1, $b \neq -\frac{8}{9}a^2$; case 2, $b = -\frac{8}{9}a^2$. In case 1, R and T admit the following expansions:

$$R = \sum_{j=0}^{\infty} \sum_{k=0}^{[j/3]} R_{j,k}(t) X^{j-1} (\ln X)^k \tag{5.3a}$$

$$T = A \ln X + \sum_{j=0}^{\infty} \sum_{k=0}^{[j/3]} T_{j,k}(t) X^{j} (\ln X)^k \tag{5.3b}$$

where, as before,

$$X := x - f(t). \tag{5.4}$$

The resonances are given by

$$(j+1)j(j-3)(j-4) = 0 \qquad j \geqslant 1 \tag{5.5}$$

and the four arbitrary functions are

$$f(t) \quad T_{0,0}(t) \qquad R_{3,0}(t) \quad \text{or} \quad T_{3,0}(t) \qquad R_{4,0}(t) \quad \text{or} \quad T_{4,0}(t). \tag{5.6}$$

The first few coefficients are

$$R_{0,0} = \pm(-\tfrac{4}{9}a^2 - \tfrac{1}{2}b)^{-1/2} \qquad A = -\tfrac{4}{3}aR_{0,0} \tag{5.7a}$$

$$R_{1,0} = -\tfrac{1}{8}AR_{0,0}f' \qquad T_{1,0} = -\tfrac{1}{8}(A^2+4)f' \tag{5.7b}$$

and the first non-trivial logarithmic terms have coefficients

$$R_{3,1} = \tfrac{1}{64}A^2R_{0,0}f'' \qquad T_{3,1} = \tfrac{1}{192}A(A^2-8)f''. \tag{5.8}$$

Note that the resonance at $j=4$ does not create any $(\ln X)^2$ terms; the first such terms appear at $j=6$.

In case 2, where $b = -\frac{8}{9}a^2$, R and T have the expansions

$$R = \sum_{j=0}^{\infty} \sum_{k=0}^{[j/2]} R_{j,k}(t) X^{j-2} (\ln X)^k \tag{5.9a}$$

$$T = T_{0,0}(t)X^{-1} + T_{1,0}(t) + T_{1,1}(t)\ln X + \sum_{j=2}^{\infty} \sum_{k=0}^{[j/2]} T_{j,k}(t) X^{j-1} (\ln X)^k \tag{5.9b}$$

in which the resonances are given by

$$(j+1)(j-1)(j-4)(j-6) = 0 \qquad j \geqslant 3. \tag{5.10}$$

This case is more difficult to treat since one must write down six recursion relations for $R_{j,k}$, $R_{j+1,k}$, $R_{j+2,k}$, $T_{j,k}$, $T_{j+1,k}$ and $T_{j+2,k}$ in order to be able to solve for $R_{j,k}$ and $T_{j,k}$ (the 6×6 coefficient matrix has rank 4 for generic j; the resonances are the values of j such that the rank is 3 or lower); the difficulty is partly alleviated by the change of variable: $\tilde{T} = T + (4a/3)\int R\,\mathrm{d}x$. The four arbitrary functions are

$$f(t) \quad T_{1,0}(t) \quad R_{4,0}(t) \quad R_{6,0}(t). \tag{5.11}$$

The first few coefficients are

$$R_{0,0}(t) = -\frac{6}{af'} \qquad T_{0,0} = -\frac{8}{f'} \qquad R_{1,0} = -\frac{4}{a}f''/(f')^3 \tag{5.12}$$

and the first logarithmic terms have coefficients

$$T_{1,1} = -\tfrac{4}{3}aR_{1,0} \qquad R_{2,1} = \frac{2}{3f'}R'_{1,0} \qquad T_{2,1} = -\frac{8a}{9f'}R'_{1,0}. \tag{5.13}$$

It is a curious fact that the resonances at $j=4$ and $j=6$ do not create any higher powers of $\ln X$ than those already caused by the logarithm at $j=1$; thus $R_{4,2}\neq 0$, $R_{6,3}\neq 0$ while, due to non-trivial cancellations, $R_{4,3}=0$, $R_{6,4}=0$, $R_{6,5}=0$ (implied by $R_{4,3}=0$), similarly for the corresponding coefficients of T.

5.2. The Wadati–Konno–Ichikawa equation

As a final application of Painlevé analysis to members of the NLS family of equations, we shall investigate the Painlevé character of the Wadati–Konno–Ichikawa (WKI) equation (Wadati *et al* 1979b, Shimizu and Wadati 1980)

$$-\mathrm{i}\,U_T=[U(1+UV)^{-1/2}]_{XX} \tag{5.14a}$$

$$\mathrm{i}\,V_T=[V(1+UV)^{-1/2}]_{XX} \tag{5.14b}$$

which is known to be integrable via its relationship to NLS (Ishimori 1982, Wadati and Sogo 1983) (see below for further details). The polar variables R and S defined by

$$U:=R(1-R^2)^{-1/2}\,\mathrm{e}^{\mathrm{i}S} \qquad V:=R(1-R^2)^{-1/2}\,\mathrm{e}^{-\mathrm{i}S} \tag{5.15}$$

have the following expansions about a movable singular hypersurface:

$$R=1+R_{1/2}(T)\,Y+R_1(T)\,Y^{3/2}+R_{3/2}(T)\,Y^2+R_2(T)\,Y^{5/2}+\ldots+R_j(T)\,Y^{j+1/2}+\ldots \tag{5.16a}$$

$$S=S_0(T)+S_{1/2}(T)\,Y^{1/2}+S_1(T)\,Y+S_{3/2}(T)\,Y^{3/2}+S_2(T)\,Y^2+\ldots+S_j(T)\,Y^j+\ldots \tag{5.16b}$$

where

$$Y:=X-F(T) \tag{5.17}$$

and j runs through all non-negative half-integer values, except that there is no $R_0(T)\,Y^{1/2}$ term. (Since the symbols X and T are used here as coordinates, the previously used X and T have been replaced by Y and S, respectively.) The resonances are given by

$$(j+\tfrac{1}{2})j(j+\tfrac{1}{2})(j-1)=0 \qquad j\geqslant 1 \tag{5.18}$$

and the four arbitrary functions are

$$F(T) \quad S_0(T) \quad S_{1/2}(T) \quad R_1(T). \tag{5.19}$$

The remaining coefficients for which $j\leqslant 1$ are given by

$$R_{1/2}(T)=-2(F')^2(S_{1/2})^{-2} \tag{5.20a}$$

$$S_1(T)=\tfrac{1}{8}(F')^{-2}[R_1(S_{1/2})^3-8F'S_0'+12(F')^2R_1(S_{1/2})^{-1}]. \tag{5.20b}$$

The movable square root singularity manifest in equation (5.16) is not one which can be removed by squaring a variable or by any simple quadratic transformation. Typically such square roots or higher algebraic singularities are created or cancelled out by transformations which mix dependent and independent variables, for example, hodograph or Legendre transformations. In the simplest case where, say, $u(x,t)$ and $U(X,T)$ satisfy PDE which are related by a Bäcklund transformation of the form

$$U=U[x,t,u(x,t)] \qquad X=X[x,t,u(x,t)] \qquad T=t \tag{5.21}$$

where both U and X are (x,t)-dependent functionals of $u(x,t)$, the $U(X,T)$ equation will have movable square roots, in general, wherever $\partial X/\partial x$ has a simple zero (which does not live on a characteristic) and movable algebraic singularities of degree $k+1$, in general, whenever $\partial X/\partial x$ has a zero of order k, provided $U[x,t,u(x,t)]$ is meromorphic in x and t in a neighbourhood of the curve on which $\partial X/\partial x=0$. The Harry–Dym equation

$$U_T=2(U^{-1/2})_{XXX} \tag{5.22}$$

(Kruskal 1975) is related by a Bäcklund transformation of this type to KdV (see, for example, Levi *et al* 1984) and more closely to MKdV (see, for example, Kawamoto 1985); in this case $\partial X/\partial x$ has double zeros at the double poles of KdV and so Harry–Dym has movable cube roots, as is easily verified directly (see Weiss 1983). In more complicated problems where more than one variable takes part in the hodograph part of the Bäcklund transformation (simplest case: put $T=T[x, t, u(x, t))]$ in (5.21)), algebraic singularities are created by the zeros of a certain Jacobian determinant; again, a zero of order k implies an algebraic singularity of degree $k+1$, in general, and the case $k=1$ may be considered generic in some sense. (The reader interested in pursuing this topic may wish to experiment with the Ampère equation (3.24).)

It is not generally recognised that the Bäcklund transformation connecting WKI to NLS is reasonably simple in both directions. Ishimori (1982) and Wadati and Sogo (1983) express the solutions $u(x, t)$ and $v(x, t)$ of NLS in terms of the solutions $U(X, T)$ and $V(X, T)$ of WKI, but do not discuss the inverse. On the other hand, Levi *et al* (1984) conjecture that the inverse can only be derived 'from a conceptual point of view' (i.e. by invoking the implicit function theorem), but that concrete construction of the inverse is only possible in the case of pure soliton solutions. Consequently, they were unable to derive any auto-Bäcklund transformations for WKI. We claim that Ishimori's WKI → NLS transformation is just a Miura transformation, where this term denotes a Bäcklund transformation between different PDE which is explicit up to quadrature in one direction and depending on a Riccati equation in the other. Indeed, by choosing the variables appropriately, the Riccati equation is the same as the one appearing in the Miura transformation connecting DNLSI (or DNLSIII) with NLS due to Wadati and Sogo (1983)— see equation (2.16*a*) above. It follows that WKI and DNLSI are related to each other by a simple transformation which is explicit up to quadratures in both directions, which we construct below, and that it is now a straightforward calculation to construct auto-Bäcklund transformations for the WKI equation. The situation is analogous to the case of the Harry–Dym equation (5.22), which is related to KdV by a Miura transformation (after appropriately arranging the transformation in Levi *et al* 1984) and to MKdV by a simple hodograph transformation and quadratures (see, for example, Kawamoto 1985).

The pair of equations (5.14*a*, *b*) is the integrability condition for an infinite number of potentials, of which we draw attention to the following six potentials, θ_1, θ_2, θ_2^*, θ_3, θ_4 and θ_4^*, defined

$$\theta_{1X} = \rho^{-5} M \tag{5.23a}$$

$$\theta_{1T} = 4\mathrm{i}\rho^{-6}(U_X V_{XX} - V_X U_{XX}) + \tfrac{1}{2}\mathrm{i}\rho^{-8}(VU_X - UV_X)M \tag{5.23b}$$

$$\theta_{2X} = (\rho U)^{-1} U_X \qquad \theta_{2T} = (\rho U)^{-1} U_T - \tfrac{1}{8}\mathrm{i}\rho^{-6} M \tag{5.24}$$

$$\theta_{2X}^* = (\rho V)^{-1} V_X \qquad \theta_{2T}^* = (\rho V)^{-1} V_T + \tfrac{1}{8}\mathrm{i}\rho^{-6} M \tag{5.25}$$

$$\theta_{3X} = \rho \qquad \theta_{3T} = \tfrac{1}{2}\mathrm{i}\rho^{-2}(VU_X - UV_X) \tag{5.26}$$

$$\theta_{4X} = U \qquad \theta_{4T} = \mathrm{i}(\rho^{-1}U)_X \tag{5.27}$$

$$\theta_{4X}^* = V \qquad \theta_{4T}^* = -\mathrm{i}(\rho^{-1}V)_X \tag{5.28}$$

where

$$\rho := (1+UV)^{1/2} \qquad M := (VU_X - UV_X)^2 - 4U_X V_X. \tag{5.29}$$

The potentials θ_2 and θ_2^* are related by

$$\theta_2+\theta_2^*=\ln\left(\frac{\rho-1}{\rho+1}\right)+\text{constant} \tag{5.30}$$

and we shall set the additive constant to be zero.

An interesting symmetry group admitted by the WKI equation, which we believe is new, is given by

$$\rho'=\Delta^{-1}\rho \qquad \Delta:=\alpha\delta+\beta\gamma-\alpha\gamma U+\beta\delta V \tag{5.31a}$$

$$U'=\Delta^{-1}(\alpha^2 U-\beta^2 V-2\alpha\beta) \tag{5.31b}$$

$$V'=\Delta^{-1}(-\gamma^2 U+\delta^2 V+2\gamma\delta) \tag{5.31c}$$

$$X'=(\alpha\delta+\beta\gamma)X-\alpha\gamma\theta_4(X,T)+\beta\delta\theta_4^*(X,T) \tag{5.31d}$$

$$T'=T \tag{5.31e}$$

where α, β, γ and δ are complex constants subject to $\alpha\delta-\beta\gamma=1$. The functions $U'(X',T')$ and $V'(X',T')$ solve (5.14a, b) with primes on all variables. The six potentials above all transform simply under this group (θ_1 and θ_3 are invariant). When all variables and parameters are complexified (which we are still assuming), the group is a two-to-one homomorphic image of SL(2, $\mathbb{C}$) and is represented by the unimodular matrix

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

If we restrict to the real slices in which X and T are real variables and U and V are complex conjugates of each other, then the star notation in (5.25) and (5.28) denotes complex conjugation and we must have

$$\delta=\alpha^* \qquad \gamma=-\beta^* \qquad \alpha\alpha^*+\beta\beta^*=1 \tag{5.32}$$

thereby restricting attention to the subgroup SU(2).

In terms of the θ potentials, Ishimori's Bäcklund transformation can be arranged in the form

$$u=\tfrac{1}{2}\mathrm{i}\frac{\partial}{\partial X}[(1+\rho^{-1})\exp\theta_2] \tag{5.33a}$$

$$v=-\tfrac{1}{2}\mathrm{i}\frac{\partial}{\partial X}[(1+\rho^{-1})\exp\theta_2^*] \tag{5.33b}$$

$$x=\theta_3(X,T) \tag{5.33c}$$

$$t=-T. \tag{5.33d}$$

The lower case variables $u(x,t)$ and $v(x,t)$ solve NLS:

$$\mathrm{i}u_t=u_{xx}+2u^2v \tag{5.34a}$$

$$-\mathrm{i}v_t=v_{xx}+2uv^2. \tag{5.34b}$$

A short calculation shows that the NLS variables u, v, x and t are invariant under the symmetry group (5.31). In terms of the NLS variables, the defining relations for θ_1 become

$$\theta_{1x}=-16uv \qquad \theta_{1t}=-16\mathrm{i}(uv_x-vu_x) \tag{5.35}$$

which shows that θ_1 is minus 16 times the ψ potential defined by equations (2.4a, b) with $a=b=0$ and u and v solving NLS; however, we caution the reader not to confuse this ψ potential with the one appearing in the next paragraph.

We now proceed to construct the inverse of the transformation (5.33) in several stages: NLS $\rightarrow$ DNLSIII $\rightarrow$ DNLSI $\rightarrow$ WKI. Let a solution (u, v) of (5.34) be given. Solve the compatible pair of Riccati equations

$$w_x = \mathrm{i}v - \mathrm{i}uw^2 \tag{5.36a}$$

$$w_t = -v_x + 2\mathrm{i}uvw - u_x w^2 \tag{5.36b}$$

for a function $w(x, t)$. The pair of variables (u, w) solve DNLSIII (2.13) with w replacing v and $b=2$. Next define

$$\tilde{u} = u\,\mathrm{e}^{2\mathrm{i}\psi} \qquad \tilde{v} = w\,\mathrm{e}^{2\mathrm{i}\psi} \tag{5.37}$$

where

$$\psi_x = uw = \tilde{u}\tilde{v} \tag{5.38a}$$

$$\psi_t = \mathrm{i}(uw_x - wu_x) - u^2w^2 = \mathrm{i}(\tilde{u}\tilde{v}_x - \tilde{v}\tilde{u}_x) + 3\tilde{u}^2\tilde{v}^2. \tag{5.38b}$$

The pair of variables $(\tilde{u}, \tilde{v})$ solve DNLSI:

$$\mathrm{i}\tilde{u}_t = \tilde{u}_{xx} + 2\mathrm{i}(\tilde{u}^2\tilde{v})_x \tag{5.39a}$$

$$-\mathrm{i}\tilde{v}_t = \tilde{v}_{xx} - 2\mathrm{i}(\tilde{u}\tilde{v}^2)_x. \tag{5.39b}$$

Equations (5.36)–(5.38) above are just the inverse of equations (2.14) and (2.16a) which define the DNLSI $\rightarrow$ NLS Miura transformation of Wadati and Sogo (1983).

To take the final step from DNLSI to the WKI equation, we need two more potentials $\tilde{w}$ and ϕ_2 which, together with ϕ_1 and ϕ_3, are defined by

$$\tilde{w}_x = \tilde{u} \qquad \tilde{w}_t = -\mathrm{i}\tilde{u}_x + 2\tilde{u}^2\tilde{v} \tag{5.40}$$

$$\phi_{1x} = \tilde{v} \qquad \phi_{1t} = \mathrm{i}\tilde{v}_x + 2\tilde{u}\tilde{v}^2 \tag{5.41}$$

$$\phi_{2x} = \mathrm{i}\tilde{v}\tilde{w} \qquad \phi_{2t} = \tilde{v}(1+2\mathrm{i}\tilde{v}\tilde{w})\tilde{w}_x - \tilde{w}\tilde{v}_x \tag{5.42}$$

$$\phi_{3x} = \tilde{w}(1+\mathrm{i}\tilde{v}\tilde{w}) \tag{5.43a}$$

$$\phi_{3t} = 2\tilde{v}\tilde{w}(1+\mathrm{i}\tilde{v}\tilde{w})\tilde{w}_x - \tilde{w}^2\tilde{v}_x - \mathrm{i}\tilde{w}_x. \tag{5.43b}$$

The DNSLI $\rightarrow$ WKI transformation is given by

$$\rho = (1+2\mathrm{i}\tilde{v}\tilde{w})^{-1} \qquad \text{(recall (5.29))} \tag{5.44a}$$

$$U = \rho\tilde{w}(1+\mathrm{i}\tilde{v}\tilde{w}) \tag{5.44b}$$

$$V = -4\mathrm{i}\rho\tilde{v} \tag{5.44c}$$

$$X = x + 2\phi_2(x, t) \tag{5.44d}$$

$$T = -t. \tag{5.44e}$$

The variables $U(X, T)$ and $V(X, T)$ solve the WKI equations (5.14a, b). The θ and ϕ potentials are related according to

$$\phi_1(x, t) = \tfrac{1}{4}\mathrm{i}\theta_4^*(X, T) \tag{5.45a}$$

$$\phi_2(x, t) = \tfrac{1}{2}X - \tfrac{1}{2}\theta_3(X, T) \tag{5.45b}$$

$$\phi_3(x, t) = \theta_4(X, T). \tag{5.45c}$$

It is practically trivial to invert equations (5.44a–e) to express the DNLSI variables $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, ψ, x and t in terms of the WKI variables U, V, ρ, X and T. Consequently, it is now an elementary problem to lift up the internal symmetry structure of NLS and DNLSI to the WKI equation. In particular, the two distinct Miura transformations connecting NLS to DNLSIII (see equations (2.16a, b)) make possible a relatively straightforward derivation of auto-Bäcklund transformations for NLS (simpler than the standard one of Lamb (1974)), the integrable members of the GDNLS and GMNLS families, and the WKI equation. These will appear in a separate paper.

The movable square roots in $U(X, T)$ and $V(X, T)$ manifest in equations (5.16a, b) occur at the simple zeros of

$$\partial X/\partial x = 1 + 2\mathrm{i}\tilde{v}\tilde{w}. \tag{5.46}$$

These singularities are not related via the Bäcklund transformation to the poles of NLS or DNLSI (unlike the case of the Harry–Dym, KdV and MKdV).

We conclude this discussion by showing how the new symmetry group (5.31) arises naturally from our interpretation of the inverse of Ishimori's transformation. Begin by expressing $u(x, t)$, $v(x, t)$, x and t in terms of the WKI variables as in equations (5.33a–d). If w is a particular solution of the Riccati equations (5.36a, b), the general solution contains one integration constant and takes the form

$$w' = w + \frac{\mathrm{i}\gamma/2}{\delta - \gamma\tilde{w}/2}\,\mathrm{e}^{-2\mathrm{i}\psi} \tag{5.47}$$

where we have chosen to write the integration constant as a ratio $\frac{1}{2}\gamma/\delta$. Next, using w' instead of w in (5.38a, b), we obtain

$$\mathrm{e}^{\mathrm{i}\psi'} = (\delta - \tfrac{1}{2}\gamma\tilde{w})\,\mathrm{e}^{\mathrm{i}\psi} \tag{5.48}$$

where now γ and δ are independent due to the additive constant in ψ'. Continuing in this manner we find

$$\tilde{u}' = (\delta - \tfrac{1}{2}\gamma\tilde{w})^{-2}\tilde{u} \tag{5.49}$$

$$\tilde{v}' = (\delta - \tfrac{1}{2}\gamma\tilde{w})[(\delta - \tfrac{1}{2}\gamma\tilde{w})\tilde{v} + \tfrac{1}{2}\mathrm{i}\gamma] \tag{5.50}$$

$$\tilde{w}' = \frac{\alpha\tilde{w} - 2\beta}{\delta - \gamma\tilde{w}/2} \qquad \alpha\delta - \beta\gamma = 1 \tag{5.51}$$

where a new additive constant β has been included in $\tilde{w}'$. Ignoring the additive constants in ϕ_1', ϕ_2' and ϕ_3', we find

$$\phi_1' = \delta^2\phi_1 + \tfrac{1}{2}\mathrm{i}\gamma\delta(x + 2\phi_2) - \tfrac{1}{4}\mathrm{i}\gamma^2\phi_3 \tag{5.52a}$$

$$\phi_2' = (\alpha\delta + \beta\gamma)\phi_2 + \beta\gamma x - 2\mathrm{i}\beta\delta\phi_1 - \tfrac{1}{2}\alpha\gamma\phi_3 \tag{5.52b}$$

$$\phi_3' = \alpha^2\phi_3 - 2\alpha\beta(x + 2\phi_2) + 4\mathrm{i}\beta^2\phi_1. \tag{5.52c}$$

Finally we replace all variables on the right-hand sides of (5.44a–e) by their primed counterparts and then express in terms of the unprimed WKI variables using (5.44a–e) (unprimed), (5.45a–c) and (5.50)–(5.52c). The result is the SL(2, $\mathbb{C}$) group of transformations (5.31).

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References

Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249-315
Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cimento* **23** 333-8
—— 1980a *J. Math. Phys.* **21** 715-21
—— 1980b *J. Math. Phys.* **21** 1006-15
Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
Chen H H, Lee Y C and Liu C S 1979 *Phys. Scr.* **20** 490-2
Dodd R K and Fordy A P 1983 *Proc. R. Soc.* A **385** 389-429
—— 1984 *J. Phys. A: Math. Gen.* **17** 3249-66
Gerdjikov V S and Ivanov M I 1982 *JINR Preprint E2-82-595* Dubna, USSR
Hastings S P and McLeod J B 1980 *Arch. Rat. Mech. Anal.* **73** 31-51
Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
Ishimori Y 1982 *J. Phys. Soc. Japan* **51** 3036-41
Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798-801
Kawata T and Inoue H 1978 *J. Phys. Soc. Japan* **44** 1968-76
Kawata T, Sakai J and Kobayashi N 1980 *J. Phys. Soc. Japan* **48** 1371-9
Kawamoto S 1985 *J. Phys. Soc. Japan* **54** 2055-6
Kruskal M D 1975 *Dynamical Systems* (*Lecture Notes in Physics* **38**) ed J Moser (Berlin: Springer) pp 310-54
—— 1982 Private communication
Kundu A 1984 *J. Math. Phys.* **25** 3433-8
Lamb G L 1974 *J. Math. Phys.* **15** 2157-65
Levi D, Ragnisco O and Sym A 1984 *Phys. Lett.* **100A** 7-10
McLeod J B and Olver P J 1983 *SIAM J. Math. Anal.* **14** 488-506
Shimizu T and Wadati M 1980 *Prog. Theor. Phys.* **63** 808-20
Wadati M, Konno K and Ichikawa Y H 1979a *J. Phys. Soc. Japan* **46** 1965-6
—— 1979b *J. Phys. Soc. Japan* **47** 1698-700
Wadati M and Sogo K 1983 *J. Phys. Soc. Japan* **52** 394-8
Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1-7
Weiss J 1983 *J. Math. Phys.* **24** 1405-13
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522-6
Zakharov V E and Shabat A B 1972 *Sov. Phys.-JETP* **34** 62-9